Bayes Estimation of Reliability for Special k-out-of-m:G Systems

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Key Words—Bayes estimator, Constant failure-rate components, Type-II censored samples.

Reader Aids—
Purpose: Report a derivation
Special math needed for explanations: Statistics
Special math needed to use results: Same
Result useful to: Statisticians, Reliability theoreticians

Abstract—Component lifetimes of a k-out-of-m:G system are s-independent and exponentially distributed. The Bayes-quadratic-loss estimator of reliability for the system is obtained using a conjugate (1-parameter case) or a noninformative (2-parameter case) prior when data are type-II censored. A simple approximation formula for the mean square error of the Bayes estimator is proposed. Based on the mean square-error criterion, the performance of the Bayes estimator under a noninformative prior is compared with that of the maximum likelihood and minimum variance s-unbiased estimators.

1. INTRODUCTION

The Bayes estimation of reliability has attracted increasing attention in the literature since the paper by Bhattacharya [2]. One reason is that this approach allows engineers to use their prior knowledge about the parameters of the failure distributions. Drake [6], Evans [7], and Crelin [5] discussed some merits of the Bayes approach in reliability context. However, almost all the previous Bayes studies concern 1-unit (1-out-of-1:G) systems except Zacks [11], who derived the Bayes estimators for parallel (1-out-of-m:G) and series (m-out-of-m:G) systems of s-independent exponential (constant failure rate) components. The general construction of the MLE and MVUE of reliability for k-out-of-m:G systems under the exponential model has been provided by Basu & El Mawaziny [1]. Based on simulated MSEs, they compared the performances of the MLE and MVUE. Chao [3] later obtained simple and satisfactory approximation formulas for MSEs of the MLE and MVUE. This paper develops the Bayes estimator for k-out-of-m:G systems with exponential components. The 1-parameter and 2-parameter models are separately considered in section 2. In the 1-parameter case, a simple approximation formula is proposed for the MSE of the Bayes estimator. Section 3 compares the three estimators (BQLE, MLE, MVUE) using the MSE criterion.

NOTATION

θ^{-1} failure rate
μ location parameter (or guarantee time)
n sample size
r specified number of failures
t mission time
λ $(t - \mu)/\theta$
$R(t)$ reliability at time t of a k-out-of-m:G system
$Y_1 < Y_2 < \ldots < Y_r$ ordered observed lifetimes
T_r $\Sigma_{i=1}^{r} Y_i + (n - r)Y_r$, total lifetimes
U_r $\Sigma_{i=1}^{r} (Y_i - Y_1) + (n - r)(Y_r - Y_1)$
ν, τ parameters in the prior distribution
C_r $(1 + nY_1/U_r)^{r+\nu-1}/[(1 + nY_1/U_r)^{r+\nu-1} - 1]$
b^+ $\max\{0, b\}$
$BQLE(\hat{R}_B)$, MLE($\hat{R}_M$), MVUE($\hat{R}_V$) Bayes-quadratic-loss estimator, maximum likelihood estimator, minimum variance s-unbiased estimator
MSE mean square error
RMSE root mean square error
$h(j,\alpha,\ell,\beta)$ $(-1)^{\alpha+\beta}\binom{m}{j}\binom{m}{\ell}\binom{m-j}{\alpha}\binom{m-\ell}{\beta}(j + \alpha)(\ell + \beta)\lambda^2 \exp[-\lambda(j + \alpha + \ell + \beta)]$

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

2. THE MODEL AND BAYES ESTIMATOR

Assumptions:

1. Component lifetimes are s-independent and each has a constant failure rate θ^{-1}, location parameter μ.
2. The n_i replicates of component i are life tested and the test is terminated at cumulative failure r of these $n = \Sigma_{i=1}^{m} n_i$ tested components, $r \le n$, r fixed.
3. In the 1-parameter case ($\mu \equiv 0$), a conjugate prior for θ is assumed; In the 2-parameter case, we use an improper noninformative prior for μ and θ.

2.1 1-Parameter case ($\mu \equiv 0$)

For the 1-unit system, Bayes estimation was first discussed in Bhattacharya [2]. Zacks [11] obtained the Bayes estimator of series and parallel systems. Consider a general k-out-of-m:G system, the reliability of the system at time t is

$$R(t) = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha}\binom{m}{j}\binom{m-j}{\alpha} \exp[-(j + \alpha)t/\theta]. \quad (1)$$

The conjugate prior distribution for θ is:

$$g(\theta) \propto \theta^{-(\nu+1)} \exp(-\tau/\theta), \ \nu, \tau \ge 0,$$

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The BQLE is:

$$\hat{R}_B = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \left[1 + \frac{(j+\alpha)t}{T_r + \tau}\right]^{-(r+\nu)}. \quad (2)$$

In the next section, we investigate the performance of the Bayes estimator under a noninformative prior. This is equivalent to $\tau \to 0$, $\nu \to 0$ in (2).

Applying the same method as in [3] and using the moments of T_r, we obtain a simple and satisfactory approximate formula for Bayes estimator under a noninformative prior:

$$\text{MSE}\{\hat{R}_B\} = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} \sum_{\ell=k}^{m} \sum_{\beta=0}^{m-\ell} h(j, \alpha, \ell, \beta) \cdot \{r^{-1} + r^{-2}[5 - 5(j+\alpha+\ell+\beta)\lambda + [(j+\alpha)^2 + (\ell+\beta)^2]\lambda^2 + (3/2)(j+\alpha)(\ell+\beta)\lambda^2]\} + O(r^{-3}). \quad (3)$$

The Bayes estimator for non-identical exponentially distributed component lifetimes can be similarly obtained by a straightforward extension.

2.2 2-Parameter case

We consider the case that $t \geq \mu$, otherwise there will be no failure. The reliability for a k-out-of-m:G system at time t when $t \geq \mu$ is

$$R(t) = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \exp\{-(j+\alpha)(t-\mu)/\theta\}. \quad (4)$$

The previous works for 1-unit systems contain references [8] - [10]. We assume here an improper noninformative prior:

$$g(\mu, \theta) \propto \theta^{-(\nu+1)}, \ \nu \geq 0.$$

The BQLE of a k-out-of-m system is:

$$\hat{R}_B = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \frac{nC_r}{n+j+\alpha} \cdot \left\{\left(1 + \frac{(j+\alpha)(t-Y_1)}{U_r}\right)^{-(r+\nu-1)} - \left(1 + \frac{nY_1 + (j+\alpha)t}{U_r}\right)^{-(r+\nu-1)}\right\}, t \geq Y_1. \quad (5)$$

When $k = m = 1$, then (5) reduces to the result provided in Sinha & Guttman [9].

3. COMPARISON

We compare the MLE, MVUE, and BLQE under a noninformative prior. Table 1 shows the estimators and their corresponding approximate MSEs.

TABLE 1
Estimators Used for Comparison and Their Approximate MSEs

	1-parameter case (equation number)	2-parameter case (equation number)
$\hat{R}_M$	(6)	(10)
$\hat{R}_V$	(7)	(11)
$\hat{R}_B$	(2)	(5)
MSE$\{\hat{R}_M\}$	(8)	(3.2) in [4]
MSE$\{\hat{R}_V\}$	(9)	(3.1) in [4]
MSE$\{\hat{R}_B\}$	(3)	

3.1 1-Parameter case

The MLE and MVUE of $R(t)$ are [1]:

$$\hat{R}_M = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \exp\{-t(j+\alpha)r/T_r\}, \quad (6)$$

$$\hat{R}_V = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \{[1-(j+\alpha)t/T_r]^{+}\}^{r-1}. \quad (7)$$

The approximate MSEs are [3]:

$$\text{MSE}\{\hat{R}_M\} = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} \sum_{\ell=k}^{m} \sum_{\beta=0}^{m-\ell} h(j, \alpha, \ell, \beta)\{r^{-1} + r^{-2}[5 - \tfrac{7}{2}(j+\alpha+\ell+\beta)\lambda + \tfrac{1}{2}[(j+\alpha)^2 + (\ell+\beta)^2]\lambda^2 + \tfrac{3}{4}(j+\alpha)(\ell+\beta)\lambda^2]\} + O(r^{-3}), \quad (8)$$

$$\text{MSE}\{\hat{R}_V\} = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} \sum_{\ell=k}^{m} \sum_{\beta=0}^{m-\ell} h(j, \alpha, \ell, \beta)\{r^{-1} + r^{-2}[2 - (j+\alpha+\ell+\beta)\lambda + \tfrac{1}{2}(j+\alpha)(\ell+\beta)\lambda^2]\} + O(r^{-3}). \quad (9)$$

Using (3) and (8), and (9), the intervals of λ, for which a given estimator has smallest MSE are found. It is well known that system reliability is a strictly decreasing function of λ. Thus, we can also find the range of system reliability for which each given estimator has smallest MSE. Such intervals are shown in table 2 for k-out-of-m:G systems, $m = 1, 2, 3$. Those tabulated intervals are independent of r. To check further the performance of (3), we also did a Monte Carlo simulation. 1000 complete samples of size 20 were generated from a population with $\theta = 1$. From each of these 1000 samples, the three estimates were obtained for given t. Finally, the sample MSE and

s-bias based on 1000 estimates for each method of estimation were calculated. The simulation results indicate that the magnitude of square s-bias for all estimators is negligible relative to the MSE. Table 2 also lists the comparison intervals using simulated MSEs. It shows that both results agree quite favorably. Generally, the Bayes estimator has smallest MSE when the system reliability is low to moderate; the MVUE is preferred when the system reliability is very low or is moderate to high; the MLE only applies if the system reliability is less than 0.20. The behaviors of RMSE vs. system reliability for some systems are plotted in figure 1.

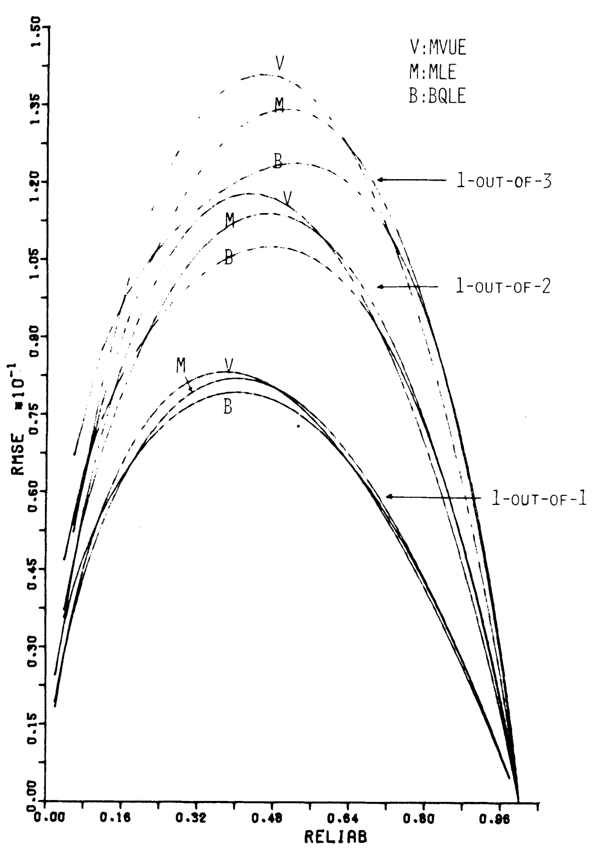

Fig. 1. Behaviors of RMSE vs. system reliability using (3), (8) and (9)

3.2 2-Parameter case

The MLE and MVUE are [4]:

$$\hat{R}_M = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} \cdot \exp[-r(j+\alpha)(t-Y_1)/U_r],\ t \ge Y_1; \tag{10}$$

$$\hat{R}_V = \sum_{j=k}^{m} \sum_{\alpha=0}^{m-j} (-1)^{\alpha} \binom{m}{j} \binom{m-j}{\alpha} [1-(j+\alpha)/n] \cdot \{[1-(j+\alpha)(t-Y_1)/U_r]^{+}\}^{r-2},\ t \ge Y_1. \tag{11}$$

TABLE 2
Range of System Reliability for Which a Given Estimator Has Smallest MSE, 1-Parameter Case, $r = n = 20$

k	m	method	MLE	BQLE	MVUE
1	1	Simu.*	(.03, .18)	(.18, .66)	(0, .03), (.66, 1)
		appr.**	(.04, .18)	(.18, .63)	(0, .04), (.63, 1)
1	2	Simu.	(.05, .22)	(.22, .73)	(0, .05), (.73, 1)
		appr.	(.06, .21)	(.21, .70)	(0, .06), (.70, 1)
2	2	Simu.	(.03, .19)	(.19, .68)	(0, .03), (.68, 1)
		appr.	(.04, .18)	(.18, .63)	(0, .04), (.63, 1)
1	3	Simu.	(.07, .23)	(.23, .76)	(0, .07), (.76, 1)
		appr.	(.07, .23)	(.23, .72)	(0, .07), (.72, 1)
2	3	Simu.	(.05, .22)	(.22, .72)	(0, .05), (.72, 1)
		appr.	(.06, .20)	(.20, .68)	(0, .06), (.68, 1)
3	3	Simu.	(.03, .18)	(.18, .68)	(0, .03), (.68, 1)
		appr.	(.04, .18)	(.18, .63)	(0, .04), (.63, 1)

* Based on simulation results, 1000 samples were generated from distribution with $\theta = 1$.

** Based on approximate MSEs, (3), (8), and (9).

Although the approximate MSE's of the MLE and MUVE can be evaluated [4], it seems difficult to approximate the MSE of the Bayes estimator. Computer simulation results are then employed to assess the relative merits of these estimators. We generated 1000 complete samples from a 2-parameter population with $\mu = 1, \theta = 1$, and $\nu = 0$ in the prior distribution. The Monte Carlo results show that for a fixed sample size, the MSEs depend only on $(t - \mu)/\theta$. Table 3 lists the intervals of system reliability corresponding to each estimator which has smallest MSE for sample size $r = n = 20$. The intervals are generally consistent with those in 1-parameter case except that the MLE is preferred in the 2-parameter case when the reliability is extremely high.

TABLE 3
Range of System Reliability for Which a Given Estimator Has Smallest MSE, 2-Parameter Case, $r = n = 20, \theta = \mu = 1, \nu = 0$

k	m	MLE	BQLE	MVUE
1	1	(.01, .15), (.94, 1)	(.15, .63)	(0, .01), (.63, .94)
1	2	(.01, .20), (.98, 1)	(.20, .74)	(0, .02), (.74, .98)
2	2	(.02, .11), (.89, 1)	(.11, .55), (.83, .89)	(0, .02), (.55, .83)
1	3	(.02, .24), (.99, 1)	(.24, .77)	(0, .02), (.77, .99)
2	3	(.02, .18), (.95, 1)	(.18, .74)	(0, .02), (.74, .95)
3	3	(.82, 1)	(.12, .43), (.70, .82)	(0, .12), (.43, .70)

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Correspondence items are not formally refereed.

Addition to: Percentiles of Pooled Estimates of Weibull Parameters

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Key Words—**Pooled estimation, Weibull distribution**

I regret that a proper acknowledgement in my recent paper [1] was not made to the work of John I. McCool [2, 3]. His work does relate, and in some cases, precede mine. Unfortunately, due to the vagaries of the mails and journal publication dates, the acknowledgement did not appear in my paper. I apologize for any inconvenience to Mr. McCool.

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