A Modified Monte Carlo Technique for Confidence Limits of System Reliability Using Pass-Fail Data

Anne Chao
National Tsing Hua University, Hsinchu
L-C Huwang
Cornell University, Ithaca

Key Words—**Binomial distribution, Confidence interval, Monte Carlo simulation, Bootstrap.**

Reader Aids—
Purpose: Improve a previous method
Special math needed for explanations: Statistics
Special math needed to use results: Same
Results useful to: Reliability analysts and theoreticians.

Abstract—**A Monte Carlo technique of Rice & Moore is modified to give a more accurate procedure for approximating confidence limits on system reliability using binomial component data. The modified lower confidence limit not only improves the Rice & Moore results but also compares well with most of the currently available methods for special series systems. Simulation results show that the modified method performs better in maintaining the nominal coverage probabilities than the original Rice & Moore procedure.**

1. INTRODUCTION

Based on component pass-fail data, Rice & Moore [12] proposed an interesting method for approximating *s*-confidence limits for system reliability under a binomial failure model. The method has the following advantages: 1) it is simple to understand, easy to implement, and quite accurate; 2) it applies to any system configuration with *s*-independent components; 3) it can handle samples with one or more components having no failures. They also numerically compared their results with results from other methods and checked the accuracy of their method through Monte Carlo simulation. However, they indicated that their estimates of the lower *s*-confidence limit usually tend to be higher than the exact values, ie, upward *s*-bias might exist. Thus, the coverage probabilities of their intervals generally are lower than the anticipated *s*-confidence levels. In this paper, we modify their method and show that the *s*-bias can be mostly eliminated while retaining all the advantages of the original procedure. Section 2 describes the modified approach and section 3 provides some numerical comparisons.

Notation

LCL	lower *s*-confidence limit
k	number of components
i	component name, $i = 1, \ldots, k$ unless otherwise stated
p_i, q_i	reliability and unreliability of component i; $p_i + q_i \equiv 1$
$h(\mathbf{p})$	$h(p_1, p_2, \ldots, p_k)$, system reliability
n_i	number of trials of component i
f_i	number of failures of component i
α, β	parameters of beta prior distribution
*	implies a replication
γ	*s*-confidence level
MLE	maximum likelihood estimator
MMLE	minimax/MLE, a modified estimator of MLE
BMLE	Bayes/MLE, a modified estimator of MLE
CMLE	50% *s*-confidence bound/MLE, a modified estimator of MLE
SMLE	Schafer/MLE, a modified estimator of MLE

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

Assumptions

1. Components are *s*-independent, which implies the system reliability is a function only of component reliabilities and system structure.
2. The number of failures for each component is a binomial r.v.

2. METHODS

2.1 *Rice & Moore Procedure (Asymptotic Normality & Monte Carlo Method)*

1. Define the system reliability $h(\mathbf{p})$, in terms of component reliabilities.
2. Calculate $\hat{p}_i = 1 - (f_i'/n_i)$, $\hat{q}_i \equiv 1 - \hat{p}_i$

$$f_i' \equiv \begin{cases} f_i\ , & \text{if } f_i \neq 0 \\ 0.25, & \text{if } f_i = 0 \text{ and } \gamma = 0.90 \\ 0.16, & \text{if } f_i = 0 \text{ and } \gamma = 0.95 \end{cases}$$

(f_i' is obtained from a log-gamma procedure in Gatliffe [7]).

3. Draw a r.v. Z_i from a standard *s*-normal distribution, and calculate $\hat{p}_i^* \equiv \hat{p}_i + Z_i(\hat{p}_i\hat{q}_i/n_i)^{1/2}$.
4. Calculate $h^* = h(\hat{\mathbf{p}}^*)$.
5. Repeat steps 3 and 4, for a total of 999 times, and get 999 estimates, $h_1^*, h_2^*, \ldots, h_{999}^*$.
6. Let $h_{(1)}^* \leqslant h_{(2)}^* \leqslant \ldots \leqslant h_{(999)}^*$ be the corresponding ordered statistics, then $h_{(50)}^*$ and $h_{(100)}^*$ are approximate 95% and 90% LCLs respectively.

2.2 *Our Modified Procedure (Bootstrap Method)*

Steps 2 and 3 of the Rice & Moore procedure are replaced by steps 2′ and 3′ respectively.

2′. Calculate $\hat{p}_i = 1 - (f_i + \alpha)/(n_i + \alpha + \beta)$, $\hat{q}_i \equiv 1 - \hat{p}_i$. We choose $\alpha = 0.2$, $\beta = 0$, as explained in section 2.3.

3′. Draw a r.v. f_i^* from a binomial population with parameters n_i and $\hat{q}_i$ defined in step 2′, and calculate $\hat{p}_i^* = 1 - (f_i^* + \alpha)/(n_i + \alpha + \beta)$, $\hat{q}_i^* \equiv 1 - \hat{p}_i^*$.

2.3 *Explanation of the Modifications*

2.3.1 Bayes Methods

Rice & Moore use the usual proportion as an estimator of the binomial probability; when zero failures occur, they use an equivalent number of failures (ie, f_i' in step 2 of the original method) derived from a log-gamma procedure [7]. We apply the Bayes estimator $1 - (f_i + \alpha)/(n_i + \alpha + \beta)$ based on a beta prior distribution having parameters α and β for the binomial probability q_i for all i.

To choose appropriate values of α and β, we first report a numerical result in table 1 to see how the LCL behaves as α and β vary for a "series" system with 2 *s*-independent components. The number of failures and test sizes are the ones studied in [11, 12]. We considered several combinations of α and β: ($\alpha = 0.1, \beta = 0$; $\alpha = 0.15, \beta = 0$; $\alpha = 0.2, \beta = 0$; $\alpha = \beta = 0.5$; $\alpha = \beta = 1$; $\alpha = 2.0, \beta = 1.0$). An advantage of the Bayes estimator is that it automatically applies to zero-failure samples. There are, of course, other estimators available for zero-failure samples.

2.3.2. Comparison with Other Methods

One referee suggested applying the estimators considered in Schafer [14]. Schafer modified the usual MLE in the following ways: when $f_i \neq 0$ or$^k{}_i \neq n_i$, always use MLE $\hat{q}_i = f_i/n_i$; when $f_i = 0$ or $f_i = n_i$, the MLE is replaced by: a) minimax estimator (The combined estimator is then called MMLE), or b) Bayes estimator (BMLE), or c) 50% *s*-confidence bound estimator (CMLE), or d) Schafer estimator (SMLE). Table 1 also lists the LCLs when $\hat{q}_i$ are estimated by these modified estimators. For a fixed estimation procedure, each tabulated bootstrap LCL is obtained by averaging 100 simulation runs.

All the LCLs are then compared with the optimum LCL calculated by Lipow & Riley [8] using the idea of Buehler [1]. The LCLs obtained when $\hat{q}_i$ are estimated by BMLE, CMLE, and SMLE give approximately the same estimates but they are much higher than the optimum LCLs. Although the LCL using MMLE performs slightly better than the other three modified estimator of MLE, it still has a positive *s*-bias. In contrast, the Bayes estimators with ($\alpha = \beta = 0.5$; $\alpha = \beta = 1$; $\alpha = 2, \beta = 1$) result in consistently lower LCLs. The LCLs using the Bayes estimator when $\beta = 0$, α is between 0.1 and 0.2 generally yield the best approximate values in the sense of being closest to the optimum LCLs.

TABLE 1
Behavior of LCLs for "Series" System with 2 Components Using Different Estimators
(Number in ***Bold Italics*** Denotes the Best Approximation to the Optimum LCL of Buehler [1])

γ	n^*	f_1	f_2	OPT**	Bayes ($\alpha=0.1$, $\beta=0$)	Bayes ($\alpha=0.15$, $\beta=0$)	Bayes ($\alpha=0.2$, $\beta=0$)	Bayes ($\alpha=0.1$, $\beta=0.2$)	Bayes ($\alpha=0.5$, $\beta=0.5$)	Bayes ($\alpha=1$, $\beta=1$)	Bayes ($\alpha=2$, $\beta=1$)	MMLE	BMLE ($\alpha=0.1$, $\beta=0$)	CMLE	SMLE
.90	10	1	1	.607	.618	.611	***.604***	.625	.527	.457	.326	.628	.637	.637	.637
		1	2	.497	.522	***.499***	.481	.535	.457	.396	.284	.531	.542	.542	.542
		2	2	.445	.421	.410	.404	***.443***	.398	.348	.248	.453	.460	.458	.460
		1	4	.344	.332	.321	.313	.356	.316	.293	.209	***.350***	.355	.355	.355
		2	3	.354	***.350***	.344	.338	.365	.340	.304	.212	.363	.365	.364	.364
	20	1	2	.716	.740	***.718***	.709	.745	.686	.627	.511	.728	.759	.758	.759
		2	2	.683	.674	.670	.665	.678	.633	.586	.480	***.685***	.695	.694	.694
		1	3	.660	.671	.666	***.661***	.676	.631	.584	.478	.678	.691	.680	.686
		2	3	.622	.630	.626	***.620***	.634	.584	.546	.450	.638	.639	.639	.639
		3	3	.585	***.588***	***.582***	.574	.593	.544	.508	.421	.598	.598	.598	.598
.95	10	1	1	.548	.553	***.544***	.537	.570	.495	.431	.286	.589	.609	.596	.603
		1	2	.443	.466	.455	***.442***	.478	.409	.367	.247	.473	.483	.477	.481
		2	2	.392	***.390***	.373	.358	.404	.346	.319	.213	.405	.405	.405	.405
		1	4	.298	.277	.269	.264	***.297***	.275	.256	.178	.273	.291	.281	.286
		2	3	.304	***.303***	.295	.289	.325	.292	.274	.182	.325	.325	.325	.325
	20	1	2	.677	.706	.697	***.683***	.710	.650	.592	.480	.708	.719	.719	.719
		2	2	.643	***.643***	.632	.626	.649	.604	.554	.450	.663	.667	.667	.667
		1	3	.620	.642	.632	***.624***	.649	.601	.551	.448	.637	.661	.659	.659
		2	3	.582	.592	.588	***.583***	.557	.552	.514	.420	.602	.604	.604	.604
		3	3	.544	.551	***.546***	.538	.596	.511	.475	.391	.560	.560	.560	.560

* $n_1 = n_2 = n$

** OPT = Optimum LCL calculated in Lipow & Riley [8] using the theory of Buehler [1]

Therefore, in the following simulation we adopt $\beta = 0$ for simplicity and only consider the case $\alpha = 0.2$. We do this because our untabulated numerical results show that the LCL for $\alpha = 0.2, \beta = 0$ is better than for $\alpha = 0.1, \beta = 0$ in maintaining the anticipated *s*-confidence levels (though both results are fairly close). When $\beta = 0$ it gives an absurd estimate $\hat{p}_i = 0$ if $f_i = n_i$. However, we are mainly interested in highly or moderately reliable components, thus the event that $f_i = n_i$ is unlikely to happen in practice.

2.3.3 Binomial Sampling

Rice & Moore apply the asymptotic normality property of $\hat{p}_i$ to generate the value of $\hat{p}_i^*$. Two difficulties arise with this approach: a) the $\hat{p}_i^*$ might be greater than l; Rice & Moore use $\min\{1, \hat{p}_i^*\}$ to eliminate this problem, b) the components should be tested enough times (viz, n_i sufficiently large) to assure the validity of the approximate *s*-normality. One of the reasons why Rice & Moore obtained higher estimates is that the n_i are usually not large enough.

Our modified procedure generates samples directly from a binomial population with binomial probability equal to the Bayes estimator, so that none of the above mentioned problems arise. Our resampling procedure is exactly the bootstrap method carried out in a parametric framework introduced by Efron [3-6]. We construct LCLs by the *percentile method* based on the bootstrap distribution [6].

3. COMPARISON

3.1 *Special Series System*

Table 2 compares our modified procedure with other procedures [l, 2, 9-13] for "series" system with 2 *s*-independent components. As mentioned before, we only consider the case $\alpha = 0.2$, $\beta = 0$ in the Bayes estimation procedure and the resulting bootstrap LCL is obtained by averaging 100 simulation runs. The results of our modified procedure are generally preferable to the Rice & Moore approximations in the sense of being closer to the optimum LCLs. For these special "series" systems, the maximum likelihood and Rice & Moore procedures seem less satisfactory. Our modified procedure and the other three methods are comparable. However, our method can be easily applied to any system (eg, *k*-out-of-*n*:G system) while some others can not. The comparison results for 3-component "series" systems are quite similar, hence we do not report those simulation results.

3.2 *Coverage Probability*

To assess the accuracy of the bootstrap approximation, we investigated its empirical coverage probabilities for 2-component series system and 2-out-of-3:G system separately in tables 3 and 4. For each fixed system, 100 data sets were generated from components with known component reliabilities and specified test sizes. Since for each generated

TABLE 2

Comparison of LCLs for 2-Component "Series" System (Number in ***Bold Italics*** Denotes the Best Approximation to the Optimum LCL of Buehler [1])

γ	n^*	f_1	f_2	AN/MC	ML	LR	OPT	AO	MMLI	BOOTSTRAP ($\alpha=0.2$, $\beta=0$)
.90	10	1	1	.655	.655	.629	.607	***.606***	.585	.604
		1	2	.542	.545	.529	.497	***.493***	.489	.481
		2	2	.458	.456	.451	.445	.430	***.441***	.404
		1	4	.337	***.347***	.350	.344	.335	.318	.313
		2	3	.372	.373	.375	.354	***.353***	.362	.338
	20	1	2	.754	.756	.739	.716	.728	***.709***	***.709***
		2	2	.700	.701	***.687***	.683	.678	.669	.665
		1	3	.693	.697	.683	.660	.675	.655	***.661***
		2	3	.646	.647	.638	.622	.628	.619	***.620***
		3	3	.599	.599	.591	.585	***.582***	.570	.574
.95	10	1	1	.614	.611	.571	.548	***.552***	.530	.537
		1	2	.495	.495	.473	.443	.435	.436	***.442***
		2	2	.414	.405	.397	.392	.382	***.391***	.358
		1	4	.290	.292	***.301***	.298	.293	.271	.264
		2	3	.328	.320	.326	.304	***.307***	.315	.289
	20	1	2	.724	.728	.700	.677	.693	***.671***	.683
		2	2	.669	.670	.647	.643	***.643***	.631	.626
		1	3	.663	.665	.643	.620	.639	***.616***	***.624***
		2	3	.612	.614	.597	.582	.593	.580	***.583***
		3	3	.565	.565	.551	.544	***.548***	.532	.538

*$n_1 = n_2 = n$

AN/MC = Asymptotic Normality/Monte Carlo (Rice & Moore [12])
ML = Maximum Likelihood (Rosenblatt [13])
LR = Likelihood Ratio (Madansky [9])
OPT = Optimum LCL (Buehler [1], Lipow & Riley [8])
AO = Approximately Optimum (Mann & Grubbs [10, 11])
MMLI = Modified Maximum Likelihood (Easterling [2])

TABLE 3

Comparison of Average LCL and Coverage Probability for 2-Component Series System (Number in ***Bold Italics*** is Closer to the Nominal Level)

γ	n^*	p_1	p_2	$h(\mathbf{p})$	AN/MC method estimate	AN/MC method coverage	Bootstrap method ($\beta = 0$, $\alpha = 0.2$) estimate	Bootstrap method ($\beta = 0$, $\alpha = 0.2$) coverage
.90	20	.95	.95	.90	.812	***.89***	.778	***.89***
		.90	.90	.81	.700	.83	.666	***.89***
		.90	.80	.72	.597	.85	.573	***.92***
		.80	.80	.64	.513	.85	.487	***.91***
	50	.95	.95	.90	.848	.87	.834	***.88***
		.90	.90	.81	.741	***.88***	.717	.93
		.90	.80	.72	.628	.87	.568	***.92***
		.80	.80	.64	.538	***.87***	.464	.95
.95	20	.95	.95	.90	.784	.89	.751	***.98***
		.90	.90	.81	.670	.89	.633	***.94***
		.90	.80	.72	.564	.89	.538	***.93***
		.80	.80	.64	.481	***.91***	.452	1.00
	50	.95	.95	.90	.834	.87	.816	***.98***
		.90	.90	.81	.722	.93	.688	***.96***
		.90	.80	.72	.606	.91	.534	***.96***
		.80	.80	.64	.516	***.92***	.421	***.98***

*$n_1 = n_2 = n$

TABLE 4
Comparison of Average LCL and Coverage Probability for 2-out-of-3:G System (Number in ***Bold Italics*** is Closer to the Nominal Level)

						AN/MC method		Bootstrap method (β = 0, α = 0.2)	
γ	n*	p_1	p_2	p_3	$h(\mathbf{p})$	estimate	coverage	estimate	coverage
.90	20	.9	.9	.9	.972	.948	.75	.934	***.83***
		.9	.9	.8	.954	.917	.76	.899	***.84***
		.9	.8	.8	.928	.875	.86	.857	***.91***
		.8	.8	.8	.896	.851	.76	.832	***.88***
	50	.9	.9	.9	.972	.954	.80	.946	***.90***
		.9	.9	.8	.954	.926	.85	.909	***.91***
		.9	.8	.8	.928	.886	***.87***	.852	.96
		.8	.8	.8	.896	.838	***.85***	.783	.98
.95	20	.9	.9	.9	.972	.937	.82	.921	***.92***
		.9	.9	.8	.954	.902	.82	.883	***.94***
		.9	.8	.8	.928	.857	.92	.836	***.95***
		.8	.8	.8	.896	.830	.85	.808	***.91***
	50	.9	.9	.9	.972	.948	.89	.936	***.95***
		.9	.9	.8	.954	.917	.90	.894	***.98***
		.9	.8	.8	.928	.875	***.93***	.831	.99
		.8	.8	.8	.896	.824	***.95***	.756	1.00

* $n_1 = n_2 = n_3 = n$

sample, 999 replications are needed to obtain a LCL, it usually requires considerable computer time when the number of simulation samples become large. Therefore, we generated only 100 samples because of computer time limitation. Then for each generated data set, the LCLs using our modified procedure and the Rice & Moore procedure were calculated. Finally, the 100 LCLs were averaged to give the approximate LCL; and the proportion of times out of 100 samples that the interval [LCL, 1] covers the true reliability were recorded to approximate the coverage probability. All the computations were conducted on a CDC Cyber-840 computer using a Fortran program. The uniform random numbers were generated from the uniform multiplicative congruential generator RANF available on CDC computers. Binomial and *s*-normal variables were generated by using IMSL routines.

3.3 *Conclusion*

In general, a desirable *s*-confidence interval should have coverage probability close enough to the anticipated *s*-confidence level. Tables 3 and 4 show that the coverage probabilities of the Rice & Moore interval are usually lower than the nominal levels, which agrees with the finding of Rice & Moore [12]. The LCL for our modified procedure performs better in maintaining the nominal coverage probability, hence we recommend it for practical applications.

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AUTHORS

Anne Chao; Institute of Applied Mathematics; National Tsing Hua University; Hsin-Chu; Taiwan-R.O. CHINA.

Anne Chao: For biography, see *IEEE Trans. Reliability*, vol R-31, 1982 Oct, p 392.

L. C. Huwang; Department of Mathematics; Cornell University; Ithaca, New York 14853 USA.

L. C. Huwang received his MS degree in Applied Mathematics from National Tsing Hua University in 1985. He is now a graduate student at Cornell University. His research interests include reliability inference and statistical simulation.

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